

Local Orthogonality: a multipartite principle for correlations

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Abstract

In recent years, the use of information principles to understand quantum correlations has been very successful. Unfortunately, the programme is limited by the fact that all principles considered so far have a bipartite formulation, but intrinsically multipartite principles, yet to be discovered, are necessary for reproducing quantum correlations. In this work, we introduce the principle of Local Orthogonality, an intrinsically multipartite principle which states that events involving different outcomes of the same local measurement must be exclusive, or orthogonal. We prove that Local Orthogonality is equivalent to no-signaling in the bipartite scenario but more restrictive for more than two parties, and that despite this equivalence, some bipartite supra-quantum correlations violate Local Orthogonality when distributed among several parties. Finally, we show how its multipartite character allows revealing the non-quantumness of correlations for which any bipartite principle fails. We believe that Local Orthogonality is a crucial ingredient for understanding no-signaling and quantum correlations.

Understanding the structure of correlations within our current description of nature, based on quantum physics, is a fundamental open problem. In particular, one would like to characterize the set of quantum correlations, that is correlations which can result from local measurements on quantum states. Pioneering work by Popescu and Rohrlich showed that the no-signaling principle – that is, the impossibility of instantaneous communication – does not suffice to recover this quantum set [1]. Indeed, they provided examples of correlations between two parties compatible with the no-signaling principle but without any quantum realization. The most paradigmatic example of these supra-quantum correlations is the so-called Popescu-Rohrlich (PR) box, also studied by Tsirelson [2]. The search for better principles separating supra-quantum correlations from quantum ones, or ideally a complete characterization of the quantum set, was started.

An important boost to this search was due to Van Dam, who introduced the idea that the existence of supra-quantum correlations, while not violating the no-signaling principle, could have implausible consequences from an information processing point of view. Van Dam showed that distant parties having access to PR-boxes can render communication complexity trivial and argued that this could be a reason for the non-existence of these correlations in Nature [3]. Since then, intensive effort has been devoted to the search for information principles characterizing the set of quantum correlations, *e.g.* the aforementioned non-trivial communication complexity [3, 4], Information Causality [5], and Macroscopic Locality [6].

Recently, it has been shown that intrinsically multipartite principles are essential to characterize the set of quantum correlations. It was proven in [7] that there exist supra-quantum

correlations for three parties that cannot be detected by any bipartite principle. Unfortunately, most of the existing principles for quantum correlations are formulated in a bipartite setting and their multipartite generalization is unclear, apart from the trivial one in which the parties are split in two groups and the principle is applied to each bipartition. In this sense, note that even the no-signaling principle has a bipartite formulation in the multipartite scenario: correlations among n distant parties satisfy the no-signaling principle whenever the marginal distribution seen by a subset of the n parties is independent of the choice of measurements by the remaining ones. It is believed that non-trivial information tasks with no quantum advantage, such as the Guess-Your-Neighbor's-Input (GYNI) inequality [8] and generalizations of it [9, 10, 11], can be regarded as multipartite extensions of the no-signaling principle.

In this work, we introduce the concept of Local Orthogonality (LO), an intrinsically multipartite principle for correlations. The principle is based on the definition of a notion of orthogonality (or exclusiveness) between events involving measurement choices and results by n distant parties: we define some events to be *orthogonal*, or *exclusive*, whenever they involve different outcomes of the same local measurement by at least one of the parties. The requirement that the sum of mutually exclusive events is less than or equal to one implies a restriction on possible correlations. Next, we provide an information processing interpretation of this new principle in terms of a distributed guessing problem. We then show that LO can be seen as a multipartite generalization of the no-signaling (NS) principle: while the set of LO correlations coincides with the set of no-signaling correlations for two parties, there exist NS correlations that violate LO for more than two parties. Despite the equivalence of LO with the NS principle in the bipartite case, we show how it can be used to detect the non-quantumness of some supra-quantum, bipartite NS correlations. In particular, we prove that distributed copies of the PR-box violate LO and show how one can get very close to the quantum boundary (Tsirelson's bound) by imposing LO. Finally, we prove that the intrinsically multipartite formulation of the principle allows one to detect supra-quantum correlations for which any bipartite principle fails. In deriving all these results, we exploit a connection between LO correlations and graph theory, related to the constructions derived in [12] in the study of quantum contextuality.

Results

Definition of Local Orthogonality. We start by presenting the formal definition of the LO principle in a general Bell scenario involving n distant parties, each of them having access to a physical system. Each party can perform m different measurements on the system, getting one out of d possible outcomes. This scenario is denoted by (n, m, d) . The measurement applied by party i is denoted by x_i , and the corresponding outcome by a_i , with $i \in \{1, \dots, n\}$, $x_i \in \{0, \dots, m-1\}$, and $a_i \in \{0, \dots, d-1\}$. The correlations among the parties are described by the joint conditional probability distribution $P(a_1 \dots a_n | x_1 \dots x_n)$, representing the probability for the parties to get outcomes a_1, \dots, a_n when making measurements x_1, \dots, x_n .

The main aim of LO is to introduce a notion of exclusiveness in the space of events. Consider first two different events e_1 and e_2 given by $e_1 = (a_1 \dots a_n | x_1 \dots x_n)$ and $e_2 = (a'_1 \dots a'_n | x'_1 \dots x'_n)$. We call these two events *locally orthogonal* or simply *orthogonal*, if they involve different outputs of the same measurement by (at least) one party. That is, if for some i we have $a_i \neq a'_i$ while $x_i = x'_i$. We then call a collection of events $\{e_i\}$ *orthogonal*, or *exclusive*, if the events are pairwise orthogonal, and impose that for any set of orthogonal events, the sum of their probabilities must not be larger than one,

$$\sum_i P(e_i) \leq 1. \quad (1)$$

This requirement is the *LO principle*. To summarize: the LO principle introduces a notion

of orthogonality in the space of events and demands that the requirement (1) is satisfied for collections of orthogonal events.

The notion of orthogonality is rather natural in the case of two events, and the requirement (1) is automatically satisfied in this case. In fact, consider two LO events e_1, e_2 with $a_i \neq a'_i$ and $x_i = x'_i$ as before. These two events can be seen as different outcomes of a correlated measurement in which: (i) party i first measures x_i and announces the outcome to the other parties and (ii) the other parties apply measurements depending on this outcome, in particular they measure x_1, \dots, x_n , if the outcome is a_i , and x'_1, \dots, x'_n otherwise. The no-signaling principle along the bipartition of i versus the other parties implies that the total probability of these two events is upper bounded by one. The principle however becomes more restrictive when extended to more events, where the previous reasoning does not apply any longer. It is precisely this extended and intrinsically multipartite formulation that makes the principle non-trivial, because it involves summing probabilities conditioned on different measurements. Taken together, all the restrictions on the conditional probability distributions implied by LO define the set of LO inequalities (1) for the (n, m, d) scenario. The set of LO correlations is then the set of conditional distributions $P(a_1 \dots a_n | x_1 \dots x_n)$ satisfying all the LO inequalities.

Moreover, we can define the sets of conditional probability distributions that obey LO for a certain number of copies in the following sense. A given distribution for the (n, m, d) scenario can be thought of as provided by some device shared between the n parties each having access to one input and output of the device. If the distribution provided by the device is compatible with LO, a natural question is whether a larger distribution coming from several copies of such a device distributed among more parties necessarily satisfies LO. As we will explain below, the answer to this question is negative. That is, LO displays activation effects and, hence, we have a hierarchy of sets. The largest set in this hierarchy, denoted \mathcal{LO}^1 , is the set of all distributions in the (n, m, d) scenario which obey the LO inequalities for this scenario. Now consider k copies of a device characterized by a distribution P , distributed among kn parties, each of which has access to one input of only one device. If the distribution p^k of the kn -partite global device obeys all the LO inequalities for the scenario (kn, m, d) , we say that p satisfies \mathcal{LO}^k , and belongs to \mathcal{LO}^k . We denote by \mathcal{LO}^∞ the set of distributions for (n, m, d) which obey the LO inequalities for any number of copies.

Having stated the LO principle, our main goal is the study of the sets \mathcal{LO}^k of LO correlations. As we shall see, the LO principle turns out to be very powerful for ruling out non-quantum correlations. As in the case of contextuality [12], graph theory seems to be perfectly suited for our purposes. We consider the $(md)^n$ possible events in the (n, m, d) scenario and map them onto an *orthogonality graph* with $(md)^n$ vertices, where two vertices are connected by an edge if and only if the corresponding events are locally orthogonal. For instance, Fig. 1 shows the orthogonality graph of the $(2, 2, 2)$ scenario. In graph theory, a *clique* in a graph $G = (V, E)$ is a subset of vertices $C \subseteq V$ such that the subgraph induced by C is complete, i.e. such that all pairs of vertices in C are connected by an edge in G . A clique is *maximal* if it cannot be extended to another clique by including a new vertex. Clearly, any clique in the orthogonality graph of events gives rise to an LO inequality and vice versa. Thus, the problem “find all the optimal LO inequalities” is equivalent to “find all maximal cliques of the associated orthogonality graph”. While the problem of finding all maximal cliques of a graph is known to be NP-hard [13, 14], there exist software packages that provide the solution for small graphs. We have used these packages to derive and partly classify LO inequalities for various Bell scenarios [15, 16].

Before concluding this part, we note that exclusive disjunction, that is, the fact that the sum of probabilities of exclusive events cannot be higher than one, was also used in Ref. [12] to bound the quantum violation of non-contextuality inequalities. However, while in contextuality scenarios there is already a natural notion of exclusive disjunction, defined by events involving

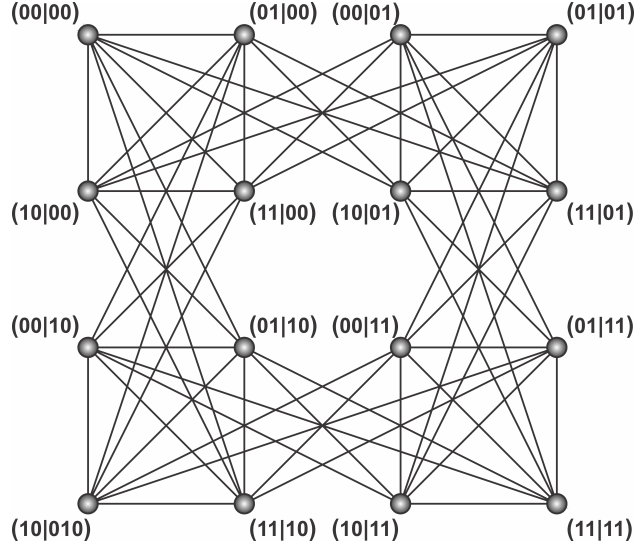


Figure 1: Orthogonality graph of the $(2,2,2)$ scenario. As mentioned in the text, each possible event corresponds to a node, while the edges connect locally orthogonal events.

only compatible tests, the relation of exclusive disjunction is no longer present in general multipartite scenarios, and it can only be applied after imposing the LO principle.

Local Orthogonality and distributed guessing problems. Before moving on to the characterization of LO correlations, we provide an interpretation of the principle from an information processing viewpoint. To this end, we introduce the notion of Distributed Guessing Problem (DGP).

Guessing problems are ubiquitous in science. The standard formulation of these problems is as follows (see Fig. 2(a)): an observer has access to some data x which depends on some initial parameter \tilde{a} , that is $x = f(\tilde{a})$. From the observed data, the observer should make a guess, a , of the initial parameter. Their goal is to maximize the probability of guessing correctly.

Guessing problems can be easily adapted to distributed scenarios. It is convenient to present the distributed guessing problem as a game and to phrase it in terms of vectors of symbols. Consider then a non-local game in which a referee has access to a set of vectors of n symbols with values in $\{0, \dots, d-1\}$. Denote this set by S and by $|S|$ its size, which can be less than d^n in general. Now, the referee chooses a vector $(\tilde{a}_1, \dots, \tilde{a}_n)$ uniformly at random from S , and encodes it into a new vector of, again, n symbols using a function f . However, the new symbols can now take m values and, thus, $f : S \rightarrow \{0, \dots, m-1\}^n$. The resulting vector is $(x_1, \dots, x_n) = f(\tilde{a}_1, \dots, \tilde{a}_n)$. These n symbols are distributed among n distant players who cannot communicate and must produce individual guesses a_1, \dots, a_n . Their goal is to guess the initial input to the function, that is, they win whenever $a_j = \tilde{a}_j$ for all j . Note that the encoding function f and the set S are known in advance to all the players.

For some f , e.g. for $x_j = \tilde{a}_j$, this task is very simple. However, there exist functions for which the task becomes extremely difficult. For a fixed size $|S|$, the most difficult functions are those for which the maximum guessing probability is equal to $1/|S|$. The players can always achieve this guessing probability by agreeing beforehand on one of the $|S|$ possible outputs, which they output regardless of the x_j . Since the input is uniform on S , their guess is correct with probability $1/|S|$. A DGP is thus maximally difficult whenever this strategy is optimal, that is, whenever it is impossible to provide a better estimate of the input than random guessing. For such an f , having access to the symbols x_j does not provide any useful information to

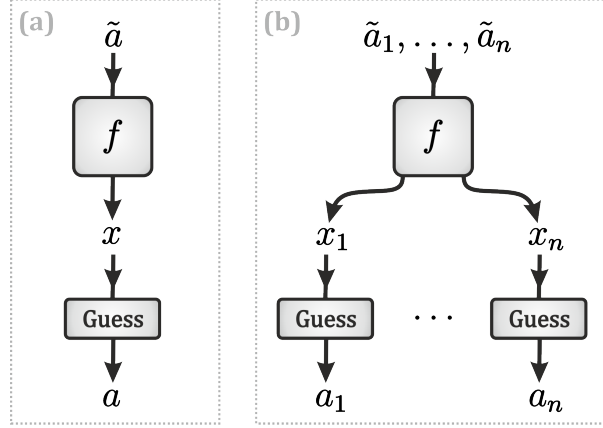


Figure 2: Guessing problems in the (a) standard and (b) distributed scenarios. In the standard scenario, an observer has to guess the value of a parameter \tilde{a} given only some function of it, $x = f(\tilde{a})$. In the distributed scenario, the input parameter is a vector of n symbols, $(\tilde{a}_1, \dots, \tilde{a}_n)$ and so is the data given to the players (x_1, \dots, x_n) . Each of player has access to just one of the symbols x_j and has to guess the corresponding initial parameter \tilde{a}_j . The game is won when all players guess correctly.

the parties. Note that non-trivial maximally difficult functions are possible only in distributed scenarios. In standard single-observer guessing problems, the only maximally difficult function is the one defined by a constant function f , which trivially erases any information about the input. An example of a difficult function in the $(n, 2, 2)$ scenario for odd n and for classically correlated players is $f(a_1, \dots, a_n) = (a_n, a_1, \dots, a_{n-1})$ defined on the set S of inputs satisfying $a_1 \oplus \dots \oplus a_n = 0$. This is the Guess-Your-Neighbor's-Input task considered in [8].

As we prove in the Methods, a DGP is maximally difficult for players sharing classical correlations if, and only if, it involves LO events. Furthermore, shared LO correlations are exactly those for which DGPs which are maximally difficult classically remain maximally difficult. Or, in other words, the existence of correlations violating LO would increase the guessing probability of some classically maximally difficult DGP. Note that, as proven below, quantum correlations satisfy LO. Thus, DGPs that are maximally difficult for classical players remain maximally difficult even for players having access to quantum correlations.

Local Orthogonality and the no-signaling principle. The first question we ask when characterizing LO correlations is how they relate to the set of no-signaling correlations, denoted by \mathcal{NS} . For bipartite scenarios, the two principles define the same set of correlations, as was already noticed in [12] (in Methods, we give a slightly different proof that emphasizes the connection with LO). However, the equivalence between LO and NS breaks down when moving to the multipartite scenario. Already in the simplest tripartite scenario $(3, 2, 2)$, we find one and only one class of non-trivial LO inequalities [17, 18], where non-trivial means that the inequalities are violated by some NS correlations. This inequality turns out to be the GYNI inequality [8], which in the tripartite case reads $P(000|000) + P(110|011) + P(011|101) + P(101|110) \leq 1$. It is easy to see by simple inspection that GYNI is an LO inequality. As shown in [8], the maximum of the GYNI inequality over \mathcal{NS} is equal to $4/3$. Our numerical data suggest that the gap between \mathcal{LO}^1 and \mathcal{NS} increases with the number of parties: in the $(4, 2, 2)$ scenario, we find 35 equivalence classes. Unfortunately, for more parties ($n > 4$), even the simplest scenario $(n, 2, 2)$ becomes computationally intractable due to the large size of the orthogonality graph. Nevertheless, examples of such inequalities for larger n as well as m and d are known and can

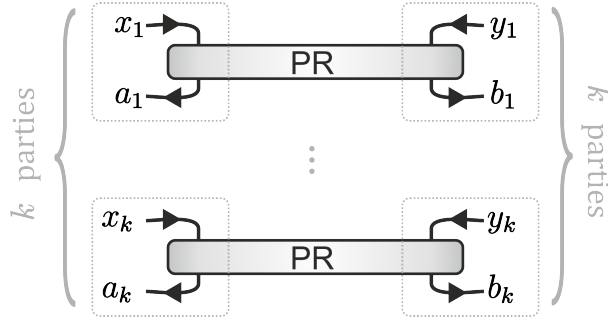


Figure 3: k copies of a PR-box shared among $2k$ parties. Each party has access to one part of a box.

be constructed from unextendible product bases [19] by using the method discussed in [9, 10] (see also [11]).

In view of these observations, LO can be interpreted as an intrinsically multipartite generalization of NS. It is worth mentioning that all the known examples of non-trivial, tight (in the sense of defining a tight Bell inequality) information tasks with no quantum advantage, given in Refs. [8, 9, 10, 11], are examples of LO inequalities. It is an interesting working conjecture to prove that any tight information task with no quantum advantage defines an LO inequality. In particular, this would imply that any tight Bell inequality in a bipartite scenario has quantum violations.

Local Orthogonality and (supra-)quantum correlations. The LO principle is naturally satisfied by quantum correlations. Indeed, orthogonal events can be associated with measurements described by projectors with disjoint supports (see Methods for details). We now investigate the use of the LO principle as a tool to detect post-quantum no-signaling correlations. Clearly, those correlations violating GYNI are in contradiction with LO as well. However, the situation turns out to be much richer, already for two parties. In principle, one might think that LO would be useless for the detection of supra-quantum bipartite correlations because of the equivalence with NS. However, this intuition is not correct. We can show that postulating LO also on the many-party level leads to detection of non-quantumness of bipartite correlations. Given some bipartite correlations, the main idea consists in distributing k copies of these among $2k$ parties, such that one party has access to one part of only one bipartite box. In the resulting $2k$ -partite scenario, the LO principle is stronger than the NS principle. Thus, it may happen that the initial bipartite correlations violate LO when distributed among different parties in a network. The idea is similar in spirit to the network approach to non-locality presented in Ref. [20].

A PR-box is a hypothetical device taking binary inputs and giving binary outputs which obey $PR(ab|xy) = 1/2$ if $a \oplus b = xy$ and $PR(ab|xy) = 0$ otherwise [1]. These boxes are known to be more non-local than what quantum theory allows. For instance, they provide a violation of the Clauser-Horne-Shimony-Holt Bell inequality [21] larger than Tsirelson's bound for quantum correlations [22]. PR-boxes are bipartite no-signaling devices, and therefore might naïvely be expected to satisfy LO. We prove now that when distributed in networks they violate LO. Consider k copies of a PR-box, distributed among $2k$ parties as shown in Fig. 3. The conditional probability distribution is:

$$P(a_1 b_1 \cdots a_k b_k | x_1 y_1 \cdots x_k y_k) = \prod_{j=1}^k PR(a_j b_j | x_j y_j), \quad (2)$$

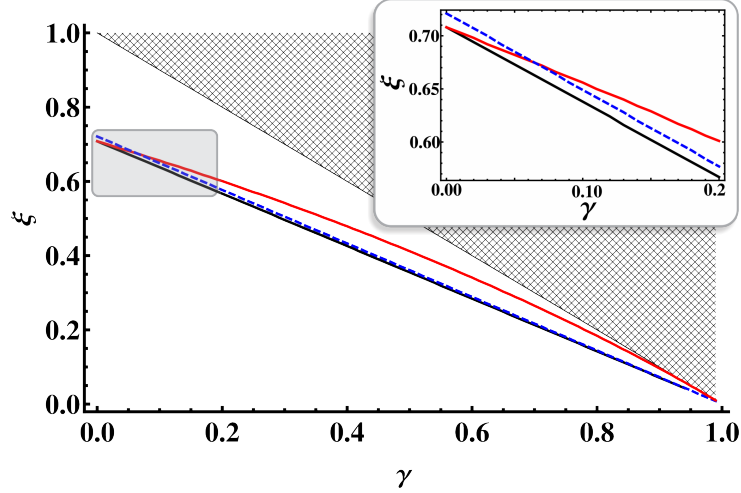


Figure 4: Comparison of IC and LO for detecting supra-quantum correlations in the $(2,2,2)$ scenario. We consider the family of correlations parametrized as $P(ab|xy) = \xi PR(ab|xy) + \gamma P_L(ab|xy) + (1 - \xi - \gamma)P_{\mathbb{I}}$, where $P_L(ab|xy) = \delta_{a,0}\delta_{b,0}$. The curves show the bounds provided by the $1+AB$ level of the NPA hierarchy [25, 26] (black), LO (blue dashed), IC (red dotted), and the edge of the crossed-out region corresponds to NS correlations and bounds the allowed parameter space. Note that when $\gamma \rightarrow 0$ (see inset), IC approximates the quantum set better than LO, which is consistent with the fact that IC recovers Tsirelson’s bound for $\gamma = 0$, while LO reaches ≈ 0.72 . LO beats the known IC bound for other parameter values.

where j labels the k boxes. Already for $k = 2$, we find LO inequalities violated by these two copies of the PR-box, and hence PR boxes do not satisfy LO². One example of such an inequality is $P(0000|0000) + P(1110|0011) + P(0011|0110) + P(1101|1011) + P(0111|1101) \leq 1$. For a PR-box, the left-hand side is equal to $5/4$. We can also analyze noisy versions of the PR-box given by $P_q := q \cdot PR + (1 - q)P_{\mathbb{I}}$, where $P_{\mathbb{I}}(ab|xy) = 1/4$ for all a, b, x, y . We find that two copies of a noisy PR-box violate LO down to $q \approx 0.72$, which is close to Tsirelson’s bound $q = 1/\sqrt{2} \approx 0.707$ (meaning that noisy boxes with $q \leq 1/\sqrt{2}$ can be simulated with quantum states and measurements). Detailed derivations of these results are presented in Methods.

An immediate consequence of these results is that LO indeed rules out all extremal boxes in the $(2, 2, d)$ scenarios. This is because using such d -outcome extremal boxes, one can always simulate correlations of the PR-box arbitrarily well [23]. More precisely, in the case of even d , a single copy of such a box is enough (one only needs to relabel the outcomes of the box), and in the case of odd d , by using enough copies of a d -outcome extremal box it is possible to simulate the PR-box with arbitrarily small error.

One proposal for a physical principle to single out quantum correlations is Information Causality (IC) [5]. A natural question then is if and when LO can do better than IC in ruling out supra-quantum correlations. Following Allcock *et al.* [24], we study LO predictions for different families of no-signaling correlations. In some situations LO provides a bound to the set of quantum correlations which is tighter than the known bounds obtained from applications of IC, as can be seen in Fig. 4.

So far we have shown how some bipartite supra-quantum correlations can be excluded by LO. Due to the intrinsically multipartite formulation of the principle, we expect it to be of particular relevance for the study of genuine multipartite correlations among more than two parties. As an example, we study how well LO performs for extremal, non-local, no-signaling correlations—or extremal NS-boxes—in the $(3, 2, 2)$ scenario. In Ref. [27], Pironio *et al.* computed all the extremal

boxes in this scenario. They can be grouped into 46 equivalence classes under symmetries, the first class corresponding to deterministic local points, while the other 45 are non-local. LO can be shown to rule out *all* these non-local extremal boxes. We follow a method similar to the one for PR-boxes presented in Methods, although for some cases the tripartite boxes required a slightly more general approach in terms of weighted graphs. We find that every non-local extremal box in $(3, 2, 2)$ violates either LO^1 or LO^2 , and hence cannot have a quantum realization.

We conclude this part with a couple of observations. First, LO is able to rule out the only extremal box in the $(2, 2, 2)$ scenario (the PR-box), as well as all extremal boxes in the $(2, 2, d)$ scenarios and, as we just saw, also in the $(3, 2, 2)$ case. To our knowledge, it is the only principle which allows doing so. Second, the intrinsically multipartite formulation of LO allows detecting correlations for which any bipartite principle fails. For instance, the correlations provided in [7] violate GYNI and, consequently, also LO, but satisfy every bipartite principle. Also box number 4 in [27] is an example of a tripartite no-signaling box which cannot be ruled out by any bipartite principle [28].

Discussion

Multipartite principles are necessary to understand the structure of quantum correlations from an information perspective. Local Orthogonality is a principle for correlations that has an intrinsically multipartite formulation. It has a natural interpretation in terms of distributed guessing problems. Moreover, it can be used in the study of correlations by applying standard techniques from graph theory. We have shown here that the principle (i) can be understood as a multipartite generalization of the no-signaling principle, (ii) gets very close to the set of quantum correlations and (iii) detects all known extremal non-local boxes as supra-quantum, including those for which any bipartite principle fails.

We believe that LO will be an essential ingredient for the characterization and understanding of correlations. An intriguing question is to understand the exact relation between this set and the set of quantum correlations. Miguel Navascués has a proof that the set of quantum correlations is strictly smaller than \mathcal{LO}^∞ [29] which will be presented in [30]. Are there simple additional principles that, together with LO, completely characterize the set of quantum correlations?

Another interesting line of research consists in extending this principle, here formulated for Bell scenarios, to other frameworks. In fact, the principle is rather general: after introducing a notion of orthogonality in a space of events, the whole machinery of graph theory and copies of the graph (networks) automatically applies. The discussion of these generalizations, in particular in relation to quantum contextuality [12], will be discussed elsewhere [30].

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Methods

Proof of the equivalence between maximally difficult DGPs and LO inequalities.

Our goal here is to prove that imposing that correlations do not provide any advantage for DGP involving maximally difficult functions f is equivalent to LO. For such an f and any correlations $P(a_1 \dots a_n | x_1 \dots x_n)$, providing no advantage for the DGP defined by f means that

$$\frac{1}{|S|} \sum_{(a_1, \dots, a_n) \in S} P(a_1 \dots a_n | f_1(a_1, \dots, a_n) \dots f_n(a_1, \dots, a_n)) \leq \frac{1}{|S|}, \quad (3)$$

where f_1, \dots, f_n refer to the components of the vector f . Note that, for simplicity, and since the goal of the parties is to provide a correct guess of the initial parameters, we slightly abuse notation and replace all \tilde{a}_j by a_j . In order to prove the correspondence, we now show that functions f that are maximally difficult for classical players are precisely those which have the property that if $f(a_1, \dots, a_n)$ and $f(a'_1, \dots, a'_n)$ are both defined, then there exists some j for which $a_j \neq a'_j$, but $f_j(a_1, \dots, a_n) = f_j(a'_1, \dots, a'_n)$. Given that f varies over all those partial functions having this property, the DGP inequalities (3) define all LO inequalities in the (n, m, d) scenario.

We first prove the only if direction by contradiction. Assume there exist a_1, \dots, a_n and a'_1, \dots, a'_n , both on which f is defined, such that for every party j , either $a_j = a'_j$ or $f_j(a_1, \dots, a_n) \neq f_j(a'_1, \dots, a'_n)$ holds true. Then the following classical strategy performs better than random guessing: for those j with $a_j = a'_j$, let them output this particular value independently of their input; for those with $a_j \neq a'_j$ and $f_j(a_1, \dots, a_n) \neq f_j(a'_1, \dots, a'_n)$, choose some function g_j such that $a_j = g_j(f_j(a_1, \dots, a_n))$ and $a'_j = g_j(f_j(a'_1, \dots, a'_n))$, and let them output $g_j(x_j)$. This strategy recovers all correct values both for the a_1, \dots, a_n as well as for the a'_1, \dots, a'_n and therefore performs better than random guessing.

Conversely, we need to show that if f has this property, then using local operations only cannot be more successful than random guessing. Thanks to convexity, it is enough to consider deterministic local strategies. If a deterministic strategy is better than random guessing, there needs to exist at least a_1, \dots, a_n and a'_1, \dots, a'_n such that the strategy works on both of these. In particular, this means that, for each party j , either $a_j = a'_j$, or party j needs to be able to tell the two cases apart via x_j , so that $f_j(a_1, \dots, a_n) \neq f_j(a'_1, \dots, a'_n)$. This implies that f cannot have the property described above.

Proof of the equivalence between NS and LO in bipartite scenarios. We prove that in the bipartite case, LO¹ and NS define the same set of correlations. Although this is already known [12], here we give a slightly different proof which emphasizes the connection with LO. To simplify the notation, in the bipartite scenario, measurements and results by the two parties are labeled by x, y and a, b , so that correlations read $P(ab|xy)$.

Let us start by characterizing the possible sets of locally orthogonal events. Recall that two events are locally orthogonal if for at least one party the settings are identical but the outcomes are different. Consider a set of locally orthogonal events which contains $(ab|xy)$ and $(a'b'|x'y)$ with $x' \neq x$. Then, this set cannot contain any event of the form $(a''b''|x''y')$ with $y' \neq y$, because it could not be locally orthogonal to both other events. From this intuition, we find that the sets of pairwise orthogonal events are either $\{(ab|x\omega_A(a)) : a, b = 0, \dots, d-1\}$ for fixed x , or $\{(ab|\omega_B(b)y) : a, b = 0, \dots, d-1\}$ for fixed y with $\omega_W : \{0, \dots, d-1\} \longrightarrow \{0, \dots, m-1\}$ ($W = A, B$) being some map.

We start by showing that sets of the first kind have the desired property; the proof for sets of the second kind is analogous. Take two such events $(ab|x\omega_A(a)) \neq (a'b'|x\omega_A(a'))$. Then either $a \neq a'$ and orthogonality holds on Alice's side, or $b \neq b'$ and orthogonality follows from Bob. This proves that we have a set of LO events. To see that the set is maximal, consider an arbitrary event $(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})$. If $\tilde{x} = x$ and $\tilde{y} = \omega_A(\tilde{a})$, then this event is already in the set. Otherwise, LO fails between $(\tilde{a}\tilde{b}|\tilde{x}\tilde{y})$ and $(\tilde{a}\tilde{b}|x\omega_A(\tilde{a}))$. Hence it is impossible to add any event to the set, so that the set is maximal.

Now we prove that every maximal LO^1 set is of one of these two forms. It is enough to show that every LO^1 set is contained in a set of this form. As noted above, for events in an LO set, one of the parties is restricted to using a single input. Hence, without loss of generality, we can take x to be fixed. Since every two orthogonal events differ on at least one outcome, there exists a function $\omega(a, b)$ such that every element in the set is of the form $(ab|x\omega(a, b))$. We complete the proof by showing that $\omega(a, b)$ does not depend on b . The existence of a and b, b' with $\omega(a, b) \neq \omega(a, b')$ would imply that $(ab|x\omega(a, b))$ and $(ab'|x\omega(a, b'))$ are not orthogonal, contradicting the assumption.

We denote by \mathcal{NS} the set of distributions $P(ab|xy)$ satisfying no-signaling. The conditions for no-signaling are

$$\sum_{b=0}^{d-1} P(ab|xy) = \sum_{b=0}^{d-1} P(ab|xy'), \quad \sum_{a=0}^{d-1} P(ab|xy) = \sum_{a=0}^{d-1} P(ab|x'y), \quad (4)$$

and we also have the normalization conditions $\sum_{a,b=0}^{d-1} P(ab|xy) = 1, \forall x, y$. We now prove that $\mathcal{LO}^1 = \mathcal{NS}$, in the present bipartite setting.

$\mathcal{LO}^1 \subseteq \mathcal{NS}$: all optimal LO^1 inequalities are of the form $\sum_{a,b=0}^{d-1} P(ab|x\omega_A(a)) \leq 1$, modulo exchanging the parties. We fix any a_0, y and y' and consider the function $\omega_A(a) = y$ if $a = a_0$ and $\omega_A(a) = y'$ if $a \neq a_0$. The LO inequality yields $\sum_{a \neq a_0, b=0}^{d-1} P(ab|xy') + \sum_{b=0}^{d-1} P(a_0b|xy) \leq 1$. Together with the normalization equation $\sum_{a,b=0}^{d-1} P(ab|xy') = 1$, this implies $\sum_{b=0}^{d-1} P(a_0b|xy) \leq \sum_{b=0}^{d-1} P(a_0b|xy')$. Since the same inequality can be derived with y and y' interchanged, we find that it actually needs to be an equality, which is (4).

$\mathcal{NS} \subseteq \mathcal{LO}^1$: start from the normalization condition $\sum_{a,b=0}^{d-1} P(ab|xy) = 1$. Using the no-signaling equations (4), we can transform it into an equality of the form $\sum_{a,b=0}^{d-1} P(ab|x\omega_A(y)) = 1$ for any given x and ω_A . It follows that $\mathcal{NS} \subseteq \mathcal{LO}^1$, and thus $\mathcal{LO}^1 = \mathcal{NS}$ in the bipartite scenario.

QM satisfies LO. It is straightforward to see that LO inequalities are satisfied by the set of quantum correlations, denoted \mathcal{Q} , that is, $\mathcal{Q} \subseteq \mathcal{LO}^\infty$. For simplicity, we give the proof for an inequality involving two LO events, but the generalization to an arbitrary LO inequality, and in particular to any number of copies, trivially follows.

Consider two LO events $e_1 = (a_1 \dots a_n | x_1 \dots x_n)$ and $e_2 = (a'_1 \dots a'_n | x'_1 \dots x'_n)$ with $a_i \neq a'_i, x_i = x'_i$, and the corresponding inequality $p(e_1) + p(e_2) \leq 1$. The maximization of the sum of these two probabilities over quantum correlations reads

$$\max_{|\Psi\rangle, \{\Pi_{a_i}^{x_i}\}} \langle \Psi | (\Pi_{a_1}^{x_1} \otimes \dots \otimes \Pi_{a_i}^{x_i} \otimes \dots \otimes \Pi_{a_n}^{x_n} + \Pi_{a'_1}^{x'_1} \otimes \dots \otimes \Pi_{a'_i}^{x'_i} \otimes \dots \otimes \Pi_{a'_n}^{x'_n}) | \Psi \rangle, \quad (5)$$

where the maximization runs over all possible states $|\Psi\rangle$ and projectors $\{\Pi_{a_i}^{x_i}\}$ acting on an arbitrary Hilbert space. Note that the maximization can be done over projective measurements without loss of generality, as the ancilla needed for a general measurement can be absorbed in the definition of the state $|\Psi\rangle$. The term in the parenthesis is equal to the sum of two orthogonal projectors, as $\Pi_{a_i}^{x_i} \Pi_{a'_i}^{x'_i} = 0$. Thus, this sum is upper bounded by the identity operator and the

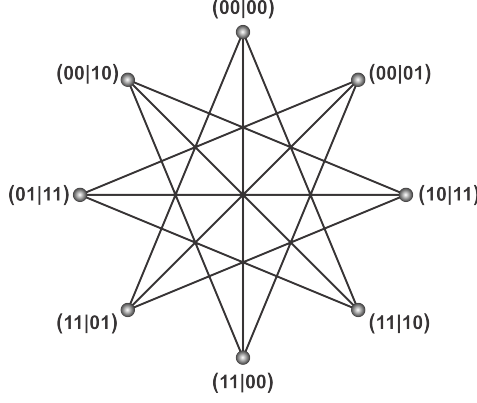


Figure 5: Orthogonality Graph of possible events for a single PR-box. It coincides with Fig. 2 in [31], where the authors study the CHSH inequality.

LO inequality follows.

Two PR-boxes violate LO. We classify events as either “possible” or “not possible” for this many-copy box as follows. An event $(ab|xy)$ for one PR-box is possible if $a \oplus b = xy$. Hence, PR-box correlations may be written as

$$PR(ab|xy) = \begin{cases} \frac{1}{2} & \text{if the event is possible,} \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

In the case of k boxes, an event $(a_1 b_1 \cdots a_k b_k | x_1 y_1 \cdots x_k y_k)$ is possible iff $a_j \oplus b_j = x_j y_j$ for all $j \in \{1, \dots, k\}$. Then, the general form for the k -box probability (2) is

$$P(a_1 b_1 \cdots a_k b_k | x_1 y_1 \cdots x_k y_k) = \begin{cases} 2^{-k} & \text{if the event is possible,} \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Consider a clique $C \subseteq V$ in the orthogonality graph $G = (V, E)$ of the scenario $(2k, 2, 2)$ and the corresponding LO inequality $LO(C)$. Define the set $C_p \subseteq C$ to be the subset of possible events in C . Then, the multipartite box (7) violates $LO(C)$ if, and only if, it violates $LO(C_p)$. In particular, in order to exclude the PR-box, it is sufficient to find a clique of size larger than 2^k in the orthogonality graph $G_{\text{poss}} = (V_p, E_p)$ of possible events for box (7). This problem becomes significantly easier, since $|V_p| = 8^k$, compared to $|V| = 16^k$ for the initial graph (compare Figs. 1 and 5). Already for $k = 2$, there exist cliques of size larger than 4. We found that all of them have size 5, and one example is given by: $\{(0000|0000), (1110|0011), (0011|0110), (1101|1011), (0111|1101)\}$.

Consider now 2 copies of a noisy PR-box given by $P_q := q \cdot PR + (1-q)P_{\mathbb{I}}$, where $P_{\mathbb{I}}(a, b|x, y) = 1/4$ for all a, b, x, y . Now all events become possible and one should consider the full list of LO inequalities for $(4, 2, 2)$. However, one can still consider this five-term inequality as associated to a non-maximal clique in the orthogonality graph of the $(4, 2, 2)$ scenario, and complete it to a maximal clique. This gives an inequality with additional terms corresponding to events that are impossible for the PR-box, but which are possible for a noisy PR-box. In this way, we have found that the distribution

$$P(a_1 b_1 a_2 b_2 | x_1 y_1 x_2 y_2) = (q \cdot PR(a_1 b_1 | x_1 y_1) + (1-q) \cdot \frac{1}{4}) \cdot (p \cdot PR(a_2 b_2 | x_2 y_2) + (1-q) \cdot \frac{1}{4}) \quad (8)$$

violates LO for $q \gtrsim 0.72$. An example of such an LO inequality is given by the following set of ten LO events: $\{(1111|0000), (1100|1010), (0100|1100), (0011|0001), (0010|0111), (1011|0000), (0101|1100), (1101|1100), (1010|0110), (1001|0100)\}$.

It appears plausible to conjecture that the generalization of the previous approach to an arbitrary number of parties converges to Tsirelson’s bound $q = 1/\sqrt{2} \approx 0.707$ in the limit of an infinite number of parties, although until now we did not find any proof.

References

- [1] S. Popescu and D. Rohrlich. Nonlocality as an axiom. *Found. Phys.*, 24:379, 1994.
- [2] B. Tsirelson. Some results and problems on quantum Bell-type inequalities. *Hadronic J. Suppl.*, 8:329–345, 1993.
- [3] W. van Dam. *Nonlocality and Communication Complexity*. PhD thesis, University of Oxford, 2000. arXiv:quant-ph/0501159 (2005).
- [4] G. Brassard, H. Buhrman, N. Linden, A. A. Méthot, A. Tapp, and F. Unger. Limit on Nonlocality in Any World in Which Communication Complexity Is Not Trivial. *Phys. Rev. Lett.*, 96:250401, Jun 2006.
- [5] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski. Information causality as a physical principle. *Nature*, 461(7267):1101–1104, October 2009.
- [6] M. Navascués and H. Wunderlich. A glance beyond the quantum model. *Proc. R. Soc. A*, 466(2115):881–890, 2010.
- [7] R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués. Quantum Correlations Require Multipartite Information Principles. *Phys. Rev. Lett.*, 107:210403, Nov 2011.
- [8] M. L. Almeida, J.-D. Bancal, N. Brunner, A. Acín, N. Gisin, and S. Pironio. Guess Your Neighbor’s Input: A Multipartite Nonlocal Game with No Quantum Advantage. *Phys. Rev. Lett.*, 104:230404, Jun 2010.
- [9] R. Augusiak, J. Stasińska, C. Hadley, J. K. Korbicz, M. Lewenstein, and A. Acín. Bell Inequalities with No Quantum Violation and Unextendable Product Bases. *Phys. Rev. Lett.*, 107:070401, Aug 2011.
- [10] R. Augusiak, T. Fritz, Ma. Kotowski, Mi. Kotowski, M. Pawłowski, M. Lewenstein, and A. Acín. Tight Bell inequalities with no quantum violation from qubit unextendible product bases. *Phys. Rev. A*, 85:042113, Apr 2012.
- [11] A. Acín, M. L. Almeida, R. Augusiak, and N. Brunner. Guess your neighbour’s input: no quantum advantage but an advantage for quantum theory. *arXiv:1205.3076*, May 2012.
- [12] A. Cabello, S. Severini, and A. Winter. (Non-)Contextuality of Physical Theories as an Axiom. *arXiv:1010.2163*, October 2010.
- [13] R. M. Karp. Reducibility Among Combinatorial Problems. In R. E. Miller and J. W. Thatcher, editors, *Complexity of Computer Computations*, pages 85–103. 1972, New York: Plenum.
- [14] In principle, it could be that, while the problem of finding the maximal cliques is NP-hard for general graphs, this is no longer the case for graphs associated to correlations among distant parties. Indeed, these graphs may represent a subset of all possible graphs that does not include the hard instances of the problem.

- [15] U. Takeaki. Mace: Maximal clique enumerator, 2005. version 2.0.
- [16] S. Niskanen and P. Östergård. Cliquer, 2010. version 1.21.
- [17] Exploiting the graph-theoretical approach mentioned in the previous section, we derive all the LO inequalities for this scenario. These inequalities are then organized into equivalence classes up to relabeling of inputs/outputs, permutations of the parties and no-signaling constraints.
- [18] To sort the inequalities into equivalence classes under relabellings, permutations of parties, and no-signalling, we used *Mathematica* as well as *MATLAB* code kindly provided to us by J. D. Bancal. This *MATLAB* code was previously used by Bancal and collaborators for classifying symmetric Bell inequalities. See <http://www.gapoptic.unige.ch/wiki/publications:attachments:bellinequalities> and references given there.
- [19] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal. Unextendible Product Bases and Bound Entanglement. *Phys. Rev. Lett.*, 82:5385–5388, 1999.
- [20] D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acín. Quantum networks reveal quantum nonlocality. *Nat. Commun.*, 2:184, February 2011.
- [21] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt. Proposed Experiment to Test Local Hidden-Variable Theories. *Phys. Rev. Lett.*, 23(15):880–884, Oct 1969.
- [22] B. S. Cirel’son. Quantum generalizations of Bell’s inequality. *Lett. Math. Phys.*, 4:93–100, 1980.
- [23] J. Barrett, N. Linden, S. Massar, S. Pironio, S. Popescu, and D. Roberts. Nonlocal correlations as an information-theoretic resource. *Phys. Rev. A*, 71:022101, 2005.
- [24] J. Allcock, N. Brunner, M. Pawłowski, and V. Scarani. Recovering part of the boundary between quantum and nonquantum correlations from information causality. *Phys. Rev. A*, 80:040103, Oct 2009.
- [25] M. Navascués, S. Pironio, and A. Acín. Bounding the set of quantum correlations. *Phys. Rev. Lett.*, 98:010401, Jan 2007.
- [26] M. Navascués, S. Pironio, and A. Acín. A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations. *New J. Phys.*, 10(7):073013, 2008.
- [27] S. Pironio, J.-D. Bancal, and V. Scarani. Extremal correlations of the tripartite no-signaling polytope. *J. Phys. A: Math. and Theor.*, 44(6):065303, 2011.
- [28] T. H. Yang, D. Cavalcanti, M. L. Almeida, C. Teo, and V. Scarani. Information-causality and extremal tripartite correlations. *New J. Phys.*, 14(1):013061, 2012.
- [29] M. Navascués, private communication.
- [30] T. Fritz, A. Leverrier, A. B. Sainz, in preparation.
- [31] M. Sadiq, P. Badziąg, M. Bourennane and A. Cabello. The most elementary Bell inequalities. *arXiv:1106.4754*, Jun 2011.